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Multi-Tethered Space-Based Interferometers: Particle System Model

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Dynamics models are presented for a class of space-based interferometers comprised of multiple component bodies, interconnected in various arrangements, by low-mass flexible tethers of variable length. The tethered constellations are to perform coordinated rotational scanning accompanied by baseline dimensional changes, as well as spin axis realignments and spin-up/spin-down maneuvers. The mechanical idealization is a system of N point masses interconnected by massless tethers of variable length. Both extensible and inextensible tethers are considered. Expressions for system angular and linear momenta are developed. The unrestricted nonlinear motion equations are derived via Lagranges equations. Rheonomic constraints are introduced to allow prescribed motion of any degrees of freedom, and the associated physical forces are determined. The linearized equations of motion are obtained for the steady rotation of a system with extensible tethers of constant unstrained length.

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Multi-Tethered Space-Based Interferometer: Particle System Model

Introduction

This report develops a dynamics model for a system of point masses interconnected by idealized tethers. The model is intended for the dynamic analysis of certain configurations of space based interferometers. The interferometer concepts under consideration are comprised of multiple collector elements and auxiliary bodies, interconnected in various arrangements by low mass flexible tethers. The constellation of components is to perform coordinated single axis rotational scanning accompanied by baseline dimensional changes. Spin axis reorientation and spin-up/spin-down maneuvers are also required. Figure-1 illustrates several candidate interferometer configurations which fall within the purview of the model developed here.

The mechanical system considered in this report is comprised of a system of point masses interconnected in an arbitrary fashion by idealized tethers. The tethers are treated as massless tensile members capable of exerting force only along the straight-line connecting the respective end masses. The tethers do not support compression or any components of shear force or bending moment. Both extensible and inextensible tethers are considered. The constitutive character for the extensible tethers is taken as visco-elastic, allowing an intrinsic energy dissipation mechanism. The amount of unstretched tether deployed between end masses is allowed to vary in a specified manner, permitting study of deployment and retraction as well as fixed length operations. Each particle is subject to an arbitrary force arising from sources external to the system. Since the system is intended to operate at the L2 Lagrange point, gravitational and other environmental forces are assumed to be of secondary importance relative to inertial effects, and hence are not specifically considered here.

The dynamical behavior of the system is described in a sequence of analyses. First, the general nonlinear unconstrained motion of the

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system is considered. Important system linear and angular momentum quantities are defined and Lagrange's equations are used to produce the motion equations. The formulation is then generalized by the addition of rheonomic constraints to allow for specification of any or all the degrees of freedom. Lagrange's equations are augmented by Lagrange multipliers to incorporate the constraints. An algorithm is provided for the solution of the constrained system that guarantees satisfaction of the constraints and determination of the associated constraint forces. The unconstrained nonlinear motion equations are then linearized by considering small amplitude motions about a nominal steady rotation. The explicit linearized equations are obtained and properties of the coefficient matrices relevant to stability analyses are established. The nonlinear dynamics of the system with inextensible tethers and prescribed motion is treated in the Appendix. Tether inextensibilty is introduced through the addition of nonlinear kinematic constraint equations. Equations are established to determine the tether tension forces and any additional external forces needed to enforce prescribed motion.

Important applications of the models developed here include: solution of particle position time histories relative to intended motions, determination of control forces necessary to produce intended motions, and evaluation of intrinsic stability characteristics of the system under conditions of steady motion. For a particle system free in space, which is intended to rotate coherently, and for which analyses of the type just mentioned are of interest, the choice of reference frame in which to describe the motion requires some consideration. Since there is no rigid body in which to embed a reference frame the usual "body fixed" frame of attitude dynamics is not available. Of course an inertial frame is available, and is indeed necessary, but coordinates referred to it are, perhaps, not the most convenient in which to linearize. A "mean axes" frame[1]* could be defined (provided the system possesses a sufficient number of particles), however, its utility is

* Superscript numbers enclosed in brackets indicate reference numbers.

diminished if small vibration modal coordinates are not introduced[2]. Given our dynamical interests, the application of a mean axes frame appears problematical. These considerations have lead to the definition of a "prescribed motion" reference frame in which to measure the particle positions. The rotation of this frame relative to inertial space can be arbitrary, but is assumed to be fully specified as a function of time. It will be found that the use of this prescribed motion frame facilitates specification of prescribed motion studies as well as linearization.

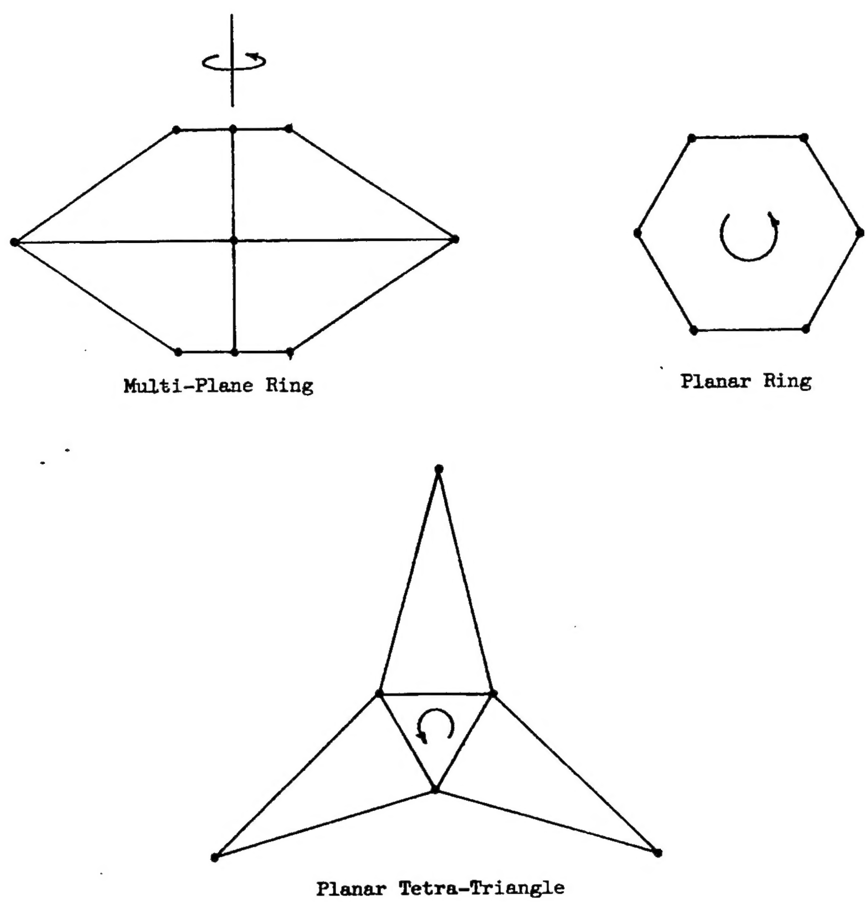

Figure-1. Tethered Interferometer Concepts within Purview of Model.

System Definition

The system is comprised of N point masses $\mathcal{P}_i$ (i=1,2,...,N). In general, $\mathcal{P}_i$ can be connected to $\mathcal{P}_j$, $(i \neq j)$, allowing the possibility of up to $\frac{1}{2}N(N-1)$ connecting tethers. Let $\mathcal{F}_I$ be an inertial reference frame, with origin at point-I, and unit basis vectors $\hat{I}$, $\hat{J}$, $\hat{K}$. $\mathcal{F}_p$ designates a "prescribed motion" reference frame. The origin of $\mathcal{F}_p$ is point-I, and its unit basis vectors are denoted $\hat{i}_p, \hat{j}_p, \hat{k}_p$. The rotational motion of $\mathcal{F}_p$ with respect to $\mathcal{F}_I$ is considered to be a known function of time. $\mathcal{P}_i$ has mass m_i, and position vector with respect to point-I given by $\bar{p}_i$. The total system mass is

$$m_t = \sum_{i=1}^{N} m_i$$

The position of the system mass center with respect to point-I is denoted $\bar{p}_\oplus$, and is defined by

$$\bar{p}_\oplus = \frac{1}{m_t}\sum_{i=1}^{N} m_i \bar{p}_i \tag{1}$$

Figure-2 illustrates the system geometry.

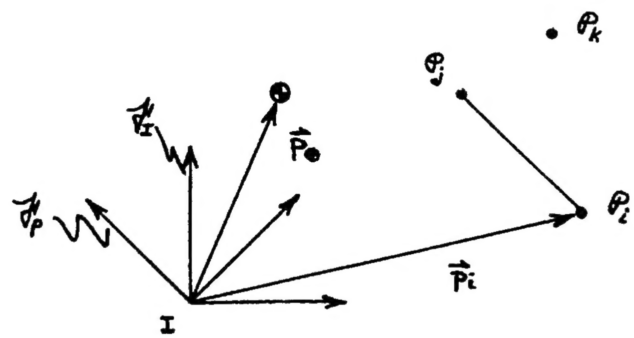

Figure-2. System Geometry.

Let $\dot{\bar{v}}$ denote the time derivative of a vector $\bar{v}$ observed in $\mathcal{F}_I$. The absolute velocity of $\mathcal{P}_i$ can then be written as

$$\dot{\bar{p}}_i = \mathring{\bar{p}}_i + \bar{\omega} \times \bar{p}_i \tag{2}$$

where $\vec{\omega}$ is the angular velocity of $\mathcal{F}_p$ with respect to $\mathcal{F}_I$, and $\overset{\circ}{\vec{p}}_i$ designates the time derivative of $\vec{p}_i$ observed in $\mathcal{F}_p$.

Of general dynamical interest are the system linear momentum and its angular momentum about point-I. The linear momentum of the system is defined as

$$\vec{L} = \sum_{i=1}^{N} \vec{L}_i = \sum_{i=1}^{N} m_i \dot{\vec{p}}_i = \sum_{i=1}^{N} m_i (\overset{\circ}{\vec{p}}_i + \vec{\omega} \times \vec{p}_i) \tag{3}$$

The angular momentum of the system about point-I is defined as

$$\vec{H} = \sum_{i=1}^{N} \vec{p}_i \times \vec{L}_i \tag{4}$$

If the vector resultant of all the external forces acting on the particles of the system is zero, then it follows from Newton's laws that

$$\dot{\vec{L}} = \vec{0} \tag{5}$$

Similarly, if the vector sum of the moments about point-I of all the external forces acting on the system is zero, then

$$\dot{\vec{H}} = \vec{0} \tag{6}$$

The kinetic energy for the system is defined as

$$T = \tfrac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{p}}_i \cdot \dot{\vec{p}}_i \tag{7}$$

Generalized Coordinates

The system configurations of interest for this report all involve a planar arrangement of collector elements that are to rotate about an axis perpendicular to that plane. Radial motions of the elements are to accompany the rotations. To ease the prescription of such motions and to facilitate later linearization, cylindrical coordinates measured relative to $\mathcal{F}_p$ are selected to describe the positions of the point masses. The cylindrical coordinates are defined such that

$$\vec{p}_i = r_i \cos\theta_i \, \hat{i}_p + r_i \sin\theta_i \, \hat{j}_p + z_i \, \hat{k}_p$$

For this choice of coordinates, it should be evident that for the study of the nominal rotational motion described above, $\hat{k}_p$ is the preferred axis of rotation. Let $\underline{p}_i$ denote the 3x1 column matrix of the scalar components of $\vec{p}_i$ resolved in $\mathcal{F}_p$. In terms of the vectrix notation of Hughes[3] we have

$$\vec{p}_i = \mathcal{F}_p^T \, \underline{p}_i = \mathcal{F}_p^T \begin{pmatrix} r_i \cos\theta_i \\ r_i \sin\theta_i \\ z_i \end{pmatrix} \tag{8}$$

The angular velocity $\vec{\omega}$ resolved into components referred to $\mathcal{F}_p$, is expressed as

$$\vec{\omega} = \mathcal{F}_p^T \, \underline{\omega} = \mathcal{F}_p^T \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

The absolute velocity of $\mathcal{P}_i$ given by Eq.(2) can now be written as

$$\dot{\vec{p}}_i = \mathcal{F}_p^T \left(\dot{\underline{p}}_i + \underline{\omega}^{\times} \underline{p}_i \right) = \mathcal{F}_p^T \begin{pmatrix} \dot{r}_i \cos\theta_i - r_i\dot{\theta}_i \sin\theta_i + \omega_y z_i - \omega_z r_i \sin\theta_i \\ \dot{r}_i \sin\theta_i + r_i\dot{\theta}_i \cos\theta_i + \omega_z r_i \cos\theta_i - \omega_x z_i \\ \dot{z}_i + \omega_x r_i \sin\theta_i - \omega_y r_i \cos\theta_i \end{pmatrix} \tag{9}$$

where we have introduced the cross product operator

$$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad \Rightarrow \quad \underline{b}^{\times} = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$$

It is desirable to refer certain vector quantities to their scalar components in $\mathcal{F}_I$. Define $[C_{Ip}]$ be the direction cosine matrix that transforms vector components referred to $\mathcal{F}_p$, to those referred to $\mathcal{F}_I$. In terms of the notation of Hughes, we write

$$\mathcal{F}_I = [C_{Ip}] \, \mathcal{F}_p \tag{10}$$

Let the vector quantities: $\vec{p}_\oplus$, $\vec{L}_i$, $\vec{L}$, and $\vec{H}$ have scalar components referred to $\mathcal{F}_p$, denoted as $\underline{p}_\oplus$, $\underline{L}_i$, $\underline{L}_p$, and $\underline{H}_p$, then we can write

$$\underline{p}_\oplus = \frac{1}{m_t}\sum_{i=1}^{N} m_i \underline{p}_i \quad ; \qquad \underline{L}_i = m_i(\dot{\underline{p}}_i + \underline{\omega}^\times \underline{p}_i)$$

$$\underline{L}_p = \sum_{i=1}^{N} \underline{L}_i \quad ; \qquad \underline{H}_p = \sum_{i=1}^{N} \underline{p}_i^\times \underline{L}_i$$

Resolving vectors $\vec{p}_\oplus$, $\vec{L}$, and $\vec{H}$ into components in $\mathcal{F}_I$, and denoting the associated 3x1 column matrices by $\underline{R}_\oplus$, $\underline{L}_I$, and $\underline{H}_I$ respectively, we have

$$\underline{R}_\oplus = [C_{Ip}]\,\underline{p}_\oplus$$

$$\underline{L}_I = [C_{Ip}]\,\underline{L}_p$$

$$\underline{H}_I = [C_{Ip}]\,\underline{H}_p$$

The quantities $\underline{L}_I$ and $\underline{H}_I$ are conserved under the conditions described preceding Eqs.(5) and (6).

Substituting Eq.(9) into (7) yields the scalar form of the system kinetic energy

$$\begin{aligned} T = \tfrac{1}{2}\sum_{i=1}^{N} m_i \Big\{ & \dot{r}_i\,[\,\dot{r}_i - 2z_i(\omega_x \sin\theta_i - \omega_y \cos\theta_i)\,] \\ & + r_i\dot{\theta}_i\,[\,r_i(\dot{\theta}_i + 2\omega_z) - 2z_i(\omega_x \cos\theta_i + \omega_y \sin\theta_i)\,] \\ & + \dot{z}_i\,[\,\dot{z}_i + 2r_i(\omega_x \sin\theta_i - \omega_y \cos\theta_i)\,] \\ & + r_i^2\,[\,\omega_z^2 + (\omega_x \sin\theta_i - \omega_y \cos\theta_i)^2\,] + z_i^2(\omega_x^2 + \omega_y^2) \\ & - 2r_i z_i \omega_z(\omega_x \cos\theta_i + \omega_y \sin\theta_i)\Big\} \end{aligned} \tag{11}$$

<u>Lagrange's Equations</u>

The equations of motion are first developed assuming the 3N generalized coordinates are independent. Later, prescribed motion constraints are introduced.

Assuming the generalized coordinates r_i, θ_i, z_i, (i=1,2,...,N) are independent, Lagrange's equations for the system can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}_i}\right) - \frac{\partial T}{\partial r_i} = Q_{ri} \tag{12a}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right) - \frac{\partial T}{\partial \theta_i} = Q_{\theta i} \tag{12b}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}_i}\right) - \frac{\partial T}{\partial z_i} = Q_{zi} \tag{12c}$$

for $i=1,2,\ldots,N$. Q_{ri}, $Q_{\theta i}$, and Q_{zi} are the generalized forces associated with the indicated degrees of freedom, arising from both internal and external forces acting on the particles. Performing the indicated derivatives on the left hand side of Eqs.(12) we record the following

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}_i}\right) = m_i\ddot{r}_i - m_i(\Omega_{2i}\dot{z}_i + \dot{\Omega}_{2i}z_i) \tag{13a}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right) = m_i r_i^2\ddot{\theta}_i + m_i[r_i^2\dot{\omega}_z + 2r_i\dot{r}_i(\dot{\theta}_i + \omega_z) - \Omega_{1i}(\dot{r}_i z_i + r_i\dot{z}_i) - r_i z_i\dot{\Omega}_{1i}] \tag{13b}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}_i}\right) = m_i\ddot{z}_i + m_i(\Omega_{2i}\dot{r}_i + \dot{\Omega}_{2i}r_i) \tag{13c}$$

$$\frac{\partial T}{\partial r_i} = m_i[(r_i\dot{\theta}_i + 2r_i\omega_z - \Omega_{1i}z_i)\dot{\theta}_i + \Omega_{2i}\dot{z}_i + (\omega_z^2 + \Omega_{2i}^2)r_i - \omega_z\Omega_{1i}z_i] \tag{14a}$$

$$\frac{\partial T}{\partial \theta_i} = m_i[\Omega_{1i}(r_i\dot{z}_i - z_i\dot{r}_i + r_i^2\Omega_{2i}) + \Omega_{2i}r_i z_i(\dot{\theta}_i + \omega_z)] \tag{14b}$$

$$\frac{\partial T}{\partial z_i} = m_i[(\omega_x^2 + \omega_y^2)z_i - \Omega_{1i}r_i(\dot{\theta}_i + \omega_z) - \Omega_{2i}\dot{r}_i] \tag{14c}$$

In Eqs.(13) and (14) we have introduced the following quantities:

$$\Omega_{1i} = \omega_x \cos\theta_i + \omega_y \sin\theta_i \tag{15a}$$

$$\Omega_{2i} = \omega_x \sin\theta_i - \omega_y \cos\theta_i \tag{15b}$$

$$\alpha_{1i} = \dot{\omega}_x \cos\theta_i + \dot{\omega}_y \sin\theta_i \tag{15c}$$

$$\alpha_{2i} = \dot{\omega}_x \sin\theta_i - \dot{\omega}_y \cos\theta_i \tag{15d}$$

$$\dot{\Omega}_{1i} = \alpha_{1i} - \Omega_{2i}\dot{\theta}_i \tag{15e}$$

$$\dot{\Omega}_{2i} = \alpha_{2i} + \Omega_{1i}\dot{\theta}_i \tag{15f}$$

Generalized Forces

The forces acting on the particles may be divided into those arising from sources external to the system and those arising from the interconnections between particles. Consider first the external forces acting on $\mathcal{P}_i$. Let $\vec{F}_i$ be the resultant of the external forces acting on $\mathcal{P}_i$. It proves convenient to resolve $\vec{F}_i$ into three orthogonal components along directions that conform naturally to the cylindrical coordinates. For $\mathcal{P}_i$ we define a Cartesian reference frame $\mathcal{F}_i$, having unit basis vectors $\hat{e}_r^{(i)}, \hat{e}_t^{(i)}, \hat{e}_z^{(i)}$ where; $\hat{e}_r^{(i)}$ is directed along $(\vec{p}_i - \vec{p}_i \cdot \hat{k}_p)$, $\hat{e}_z^{(i)}$ is parallel to $\hat{k}_p$, and $\hat{e}_t^{(i)} = \hat{e}_z^{(i)} \times \hat{e}_r^{(i)}$. Then we can write

$$\vec{F}_i = \mathcal{F}_i^T \begin{pmatrix} F_{ri} \\ F_{ti} \\ F_{zi} \end{pmatrix} \tag{16}$$

Figure-3 illustrates the situation in the $\hat{i}_p\hat{j}_p$ plane.

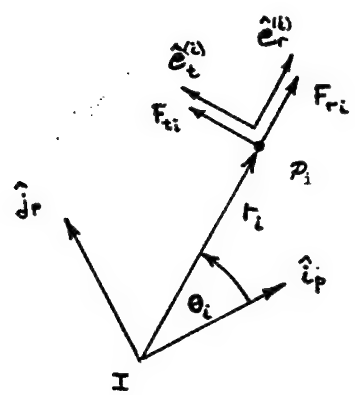

Figure-3. Resolution of External Force on $\mathcal{P}_i$.

From Eqs.(8) and (10) we have

$$\vec{p}_i = \mathcal{F}_I^T [C_{Ip}(t)] \underline{p}_i(r_i, \theta_i, z_i)$$

The virtual displacement of $\mathcal{P}_i$ is given by

$$\delta\vec{p}_i = \frac{\partial\vec{p}_i}{\partial r_i}\delta r_i + \frac{\partial\vec{p}_i}{\partial\theta_i}\delta\theta_i + \frac{\partial\vec{p}_i}{\partial z_i}\delta z_i$$

$$= \mathcal{F}_I^T\,[C_{Ip}]\left(\frac{\partial}{\partial r_i}(\underline{p}_i)\delta r_i + \frac{\partial}{\partial\theta_i}(\underline{p}_i)\delta\theta_i + \frac{\partial}{\partial z_i}(\underline{p}_i)\delta z_i\right)$$

$$= \mathcal{F}_p^T\begin{pmatrix}\cos\theta_i\ \delta r_i - r_i\sin\theta_i\ \delta\theta_i\\ \sin\theta_i\ \delta r_i + r_i\cos\theta_i\ \delta\theta_i\\ \delta z_i\end{pmatrix} \tag{17}$$

where δr_i, $\delta\theta_i$, and δz_i are the virtual displacements of the generalized coordinates of $\mathcal{P}_i$. The direction cosine matrix transforming vector components from $\mathcal{F}_p$ to $\mathcal{F}_i$ is given by

$$[\,C_{ip}] = \begin{bmatrix}\cos\theta_i & \sin\theta_i & 0\\ -\sin\theta_i & \cos\theta_i & 0\\ 0 & 0 & 1\end{bmatrix}$$

Resolved into scalar components referred to $\mathcal{F}_i$, $\delta\vec{p}_i$ can be expressed as

$$\delta\vec{p}_i = \mathcal{F}_i^T\begin{pmatrix}\delta r_i\\ r_i\delta\theta_i\\ \delta z_i\end{pmatrix}$$

The virtual work performed by the external forces acting on the system is given by

$$\delta W^{(e)} = \sum_{i=1}^{N}\vec{F}_i\cdot\delta\vec{p}_i$$

$$= \sum_{i=1}^{N}\left(F_{ri}\delta r_i + F_{ti}r_i\delta\theta_i + F_{zi}\delta z_i\right)$$

$$= \sum_{i=1}^{N}\left(Q_{ri}^{(e)}\,\delta r_i + Q_{\theta i}^{(e)}\,\delta\theta_i + Q_{zi}^{(e)}\,\delta z_i\right) \tag{18}$$

In Eq.(18) we have introduced the notation, $Q_{\beta i}^{(e)}$, to designate the generalized force associated with generalized coordinate β_i,

($\beta_i = r_i, \theta_i, z_i$) arising from the action of the external forces acting on $\mathcal{P}_i$.

Let $\vec{f}_{ij}$ be the force acting on $\mathcal{P}_i$ arising from its connection to $\mathcal{P}_j$, ($i \neq j$). To allow arbitrary connectivity we assume that each particle could be connected to every other particle of the system. The vector resultant of all internal forces acting on $\mathcal{P}_i$ is

$$\vec{f}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \vec{f}_{ij}$$

The virtual work performed by $\vec{f}_i$ on $\mathcal{P}_i$ is given by

$$\delta W_i^{(i)} = \vec{f}_i \cdot \delta\vec{p}_i = \sum_{j=1}^{N} \vec{f}_{ij} \cdot \delta\vec{p}_i$$

The virtual work performed by all the internal forces acting on all the particles of the system is

$$\delta W^{(i)} = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \vec{f}_{ij} \cdot \delta\vec{p}_i \tag{19}$$

By assumption the tethers are only capable of exerting tensile force along the straight line connecting their respective end masses, thus

$$f_{ij} = \left| \vec{f}_{ij} \right| \geq 0 \tag{20}$$

Let $\vec{l}_{ij}$ be the position vector from $\mathcal{P}_i$ to $\mathcal{P}_j$, i.e.

$$\vec{l}_{ij} = \vec{p}_j - \vec{p}_i = \mathcal{F}_p^T \begin{pmatrix} r_j \cos\theta_j - r_i \cos\theta_i \\ r_j \sin\theta_j - r_i \sin\theta_i \\ z_j - z_i \end{pmatrix} \tag{21}$$

The distance between $\mathcal{P}_i$ and $\mathcal{P}_j$ is

$$l_{ij} = l_{ji} = \left| \vec{l}_{ij} \right|$$

$$= \sqrt{r_i^2 + r_j^2 - 2 r_i r_j \cos(\theta_j - \theta_i) + (z_j - z_i)^2} \tag{22}$$

and its time derivative is

$$\dot{l}_{ij} = [r_i \dot{r}_i + r_j \dot{r}_j - (\dot{r}_i r_j + r_i \dot{r}_j)\cos(\theta_j - \theta_i)$$

$$+ r_i r_j(\dot{\theta}_j - \dot{\theta}_i)\sin(\theta_j - \theta_i) + (z_j - z_i)(\dot{z}_j - \dot{z}_i)] / l_{ij}$$

To allow the possibility of energy dissipation in the tethers, their constitutive character is assumed to be visco-elastic. Further, we assume that the length of unstrained tether deployed between any two particles of the system is a known function of time. Let the scalar $d_{ij}(t)$ be the length of unstrained tether deployed between $\mathcal{P}_i$ and $\mathcal{P}_j$ at time t. Temporarily dropping the particle subscript notation, the longitudinal strain of a tether is defined as

$$\varepsilon = \frac{l(t) - d(t)}{d(t)}$$

Assuming a Kelvin-Voigt[4] constitutive law, the tensile stress, τ, and the longitudinal strain are related by

$$\tau = E\left(\varepsilon + a\dot{\varepsilon}\right)$$

where E is the effective Young's modulus for the tether and a is a constant dissipation parameter. If A is the effective cross sectional area of the tether, then the tension is given by

$$f = \begin{cases} A\tau & \varepsilon \geq 0,\ \tau \geq 0 \\ 0 & \end{cases}$$

Assuming that $\varepsilon << 1$, and re-establishing the particle subscript notation, the tension in the tether connecting $\mathcal{P}_i$ and $\mathcal{P}_j$ can be written as

$$f_{ij} = \begin{cases} k_{ij}(l_{ij} - d_{ij}) + c_{ij}(\dot{l}_{ij} - \dot{d}_{ij}) \text{ if } \begin{cases} l_{ij} \geq d_{ij} \\ c_{ij}(\dot{l}_{ij} - \dot{d}_{ij}) \geq -k_{ij}(l_{ij} - d_{ij}) \end{cases} \\ 0 \quad \text{otherwise} \end{cases} \tag{23}$$

where the effective stiffness k_{ij}, and damping coefficient c_{ij}, are

$$k_{ij} = k_{ji} = \frac{(EA)_{ij}}{d_{ij}} \qquad ; \qquad c_{ij} = c_{ji} = a_{ij} k_{ij}$$

The virtual work done by the internal forces of the system, given by Eq. (19), can now be expressed as

$$\delta W^{(i)} = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{f_{ij}}{l_{ij}} \vec{l}_{ij} \cdot \delta \vec{p}_i$$

Using Eqs. (17) and (21) we get

$$\delta W^{(i)} = \sum_{i=1}^{N} [\, Q_{ri}^{(i)} \delta r_i + Q_{\theta i}^{(i)} \delta \theta_i + Q_{zi}^{(i)} \delta z_i \,]$$

where the generalized forces associated with the internal forces are given by

$$\underline{S}_i = \begin{pmatrix} Q_{ri}^{(i)} \\ Q_{\theta i}^{(i)} \\ Q_{zi}^{(i)} \end{pmatrix} = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{f_{ij}}{l_{ij}} \begin{pmatrix} \sigma_{ij} \\ \tau_{ij} \\ \zeta_{ij} \end{pmatrix} \tag{24}$$

In Eq. (24) we have introduced the quantities

$$\sigma_{ij} = r_j \cos(\theta_j - \theta_i) - r_i \tag{25a}$$

$$\tau_{ij} = -\tau_{ji} = r_i r_j \sin(\theta_j - \theta_i) \tag{25b}$$

$$\zeta_{ij} = -\zeta_{ji} = (z_j - z_i) \tag{25c}$$

The generalized forces appearing on the right hand side of Eq. (12) can now be written as

$$Q_{\beta i} = Q_{\beta i}^{(e)} + Q_{\beta i}^{(i)} \tag{26}$$

$(\beta_i = r_i, \theta_i, z_i)$.

Substituting Eqs. (13), (14) and (26) into Eqs. (12) yields the nonlinear motion equations for the system

$$[M]_i \underline{\ddot{u}}_i = \underline{Q}_i^{(e)} + \underline{N}_i + \underline{S}_i \tag{27}$$

$(i = 1, 2, \ldots, N)$, where

$$[M]_i = m_i \begin{bmatrix} 1 & 0 & 0 \\ 0 & r_i^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{u}_i = \begin{pmatrix} r_i \\ \theta_i \\ z_i \end{pmatrix} \quad ; \quad \underline{Q}_i^{(e)} = \begin{pmatrix} F_{ri} \\ r_i F_{ti} \\ F_{zi} \end{pmatrix}$$

$$\underline{N}_i = m_i \begin{pmatrix} r_i(\omega_z^2 + \Omega_{2i}^2) + z_i(\alpha_{2i} - \omega_z \Omega_{1i}) + r_i(\dot{\theta}_i + 2\omega_z)\dot{\theta}_i + 2\Omega_{2i}\dot{z}_i \\ r_i^2(\Omega_{1i}\Omega_{2i} - \dot{\omega}_z) + r_i z_i[\dot{\Omega}_{1i} + \Omega_{2i}(\dot{\theta}_i + \omega_z)] + 2r_i[\Omega_{1i}\dot{z}_i - (\dot{\theta}_i + \omega_z)\dot{r}_i] \\ z_i(\omega_x^2 + \omega_y^2) - 2\Omega_{2i}\dot{r}_i - r_i[\alpha_{2i} + \Omega_{1i}(2\dot{\theta}_i + \omega_z)] \end{pmatrix} \tag{28}$$

Prescribed Motion

Define the vectors of generalized coordinates and forces for the system as

$$\{q\}^T = \{r_1\ \theta_1\ z_1\ r_2\ \theta_2\ z_2\ \ldots\ r_N\ \theta_N\ z_N\} = \{q_1\ q_2\ \ldots\ q_{3N}\}$$

$$\{Q\}^T = \{Q_{r1}\ Q_{\theta 1}\ Q_{z1}\ Q_{r2}\ Q_{\theta 2}\ Q_{z2}\ \ldots\ Q_{rN}\ Q_{\theta N}\ Q_{zN}\} = \{Q_1\ Q_2\ \ldots\ Q_{3N}\}$$

Using the above notation and that of Eq.(26), Lagrange's equations for the unconstrained system can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k^{(e)} + Q_k^{(i)}$$

for $k = 1, 2, \ldots, 3N$, or, collectively, as

$$\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}}\right\} - \left\{\frac{\partial T}{\partial q}\right\} = \left\{Q^{(e)}\right\} + \left\{Q^{(i)}\right\} \tag{29}$$

For the study of prescribed motion we assume that n_p of the generalized coordinates $\{q\}$, are known functions of time. We assume the presence of rheonomic constraints of the form

$$\{q_p(t)\} = [S_p]^T\{q\} \tag{30}$$

where $\{q_p(t)\}$ is an $n_p \times 1$ column matrix of known time functions $(n_p \leq 3N)$, and $[S_p]$ is a $3N \times n_p$ Boolean selection matrix. The generalized

coordinates have been chosen so that the system motions of interest could be described in this simple manner. It is known from analytical mechanics that Lagrange's equations for a system subject to constraints of the form given by Eq.(30), can be written as

$$\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}}\right\} - \left\{\frac{\partial T}{\partial q}\right\} = \left\{Q^{(e)}\right\} + \left\{Q^{(i)}\right\} - [\,S_p\,]\{\lambda\} \tag{31}$$

where $\{\lambda\}$ is an $n_p \times 1$ vector of Lagrange multipliers.

Using the nomenclature established for Eq.(27), the nonlinear motion equations for the system subject to the constraints of Eq.(30) are now written as

$$[\,\mathcal{M}\,]\{\ddot{q}\} = \{Q^{(e)}\} + \{\mathcal{N}\} + \{\mathcal{S}\} - [S_p]\{\lambda\} \tag{32}$$

where

$$[\,\mathcal{M}\,] = \begin{bmatrix} [M]_1 & [0] & \cdots & [0] \\ [0] & [M]_2 & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots \\ [0] & [0] & \cdots & [M]_N \end{bmatrix} \quad ; \quad \{Q^{(e)}\} = \begin{pmatrix} \underline{Q}_1^{(e)} \\ \underline{Q}_2^{(e)} \\ \vdots \\ \underline{Q}_N^{(e)} \end{pmatrix}$$

$$\{\mathcal{N}\} = \begin{pmatrix} \underline{N}_1 \\ \underline{N}_2 \\ \vdots \\ \underline{N}_N \end{pmatrix} \quad ; \quad \{\mathcal{S}\} = \begin{pmatrix} \underline{S}_1 \\ \underline{S}_2 \\ \vdots \\ \underline{S}_N \end{pmatrix}$$

The last term on the right hand side of Eq.(32) can be interpreted as the generalized force required to enforce the constraints, .i.e.

$$\{Q^{(c)}\} = -[\,S_p\,]\{\lambda\} \tag{33}$$

To establish the physical forces of constraint some specific deterministic arrangement of scalar force components must be considered. Here we shall assume a system of constraint forces $\vec{F}_i^{(c)}$, $i = 1, 2, \ldots, N$, (not necessarily all nonzero) which act in parallel

with the applied external forces defined by Eq.(16). Resolved in frame $\mathcal{F}_i$ the components of $\vec{F}_i^{(c)}$ are designated as

$$\vec{F}_i^{(c)} = \mathcal{F}_i^T \begin{pmatrix} F_{ri}^{(c)} \\ F_{ti}^{(c)} \\ F_{zi}^{(c)} \end{pmatrix}$$

Equation (18) specifies the relations between the components of the external forces and the corresponding generalized forces. Relations identical to those of Eq.(18) exist between the components of $\vec{F}_i^{(c)}$ and the generalized constraint forces embodied by Eq.(33). Thus, the physical constraint forces are related to the Lagrange multipliers by

$$\begin{pmatrix} F_{r1}^{(c)} \\ r_1 F_{t1}^{(c)} \\ F_{z1}^{(c)} \\ \hdashline F_{r2}^{(c)} \\ r_2 F_{t2}^{(c)} \\ F_{z2}^{(c)} \\ \hdashline \vdots \end{pmatrix} = -[S_p]\{\lambda\} \tag{34}$$

Equations (30) and (32) constitute a system of $3N+n_p$ differential-algebraic equations in $3N+n_p$ unknowns, $\{q\}$ and $\{\lambda\}$. Given the simple form of the constraint equations, Eq.(30), a coordinate transformation can be made which yields a set of differential equations in terms of an independent set of generalized coordinates, which are uncoupled from the Lagrange multipliers. Define the set of generalized coordinates which are *not* prescribed (i.e. which are "free") as

$$\{q_f\} = [S_f]^T\{q\} \tag{35}$$

where $[S_f]$ is a $3N\times(3N-n_p)$ Boolean selection matrix. Equations (30) and (35) can be written together as

$$\begin{pmatrix} \{q_f\} \\ \{q_p\} \end{pmatrix} = \Big[\, [S_f] \;\; [S_p] \,\Big]^T\{q\} = [S]^T\{q\}$$

It follows from the simple Boolean structure of the matrices $[S_f]$ and $[S_p]$, and the complementary nature of the coordinates $\{q_f\}$ and $\{q_p\}$, that $[S]$ is an orthogonal matrix. Consequently, we have

$$\{q\} = \begin{bmatrix} [S_f] & [S_p] \end{bmatrix} \begin{pmatrix} \{q_f\} \\ \{q_p\} \end{pmatrix} \tag{36}$$

$$[S_p]^T [S_p] = [1] \quad n_p \times n_p$$

$$[S_f]^T [S_p] = [0] \quad (3N - n_p) \times n_p$$

Substituting Eq. (36) into Eq. (32), and premultiplying the result by $[S]^T$, yields the partitioned system

$$[S_f]^T [\mathcal{M}] [S_f] \{\ddot{q}_f\} = [S_f]^T \left(\{Q^{(e)}\} + \{\mathcal{N}\} + \{\mathcal{S}\} - [\mathcal{M}] [S_p] \{\ddot{q}_p\} \right) \tag{37}$$

$$\{\lambda\} = [S_p]^T \left(\{Q^{(e)}\} + \{\mathcal{N}\} + \{\mathcal{S}\} - [\mathcal{M}] \left([S_f] \{\ddot{q}_f\} + [S_p] \{\ddot{q}_p\} \right) \right) \tag{38}$$

Given $\{q_p\}$, $\{\dot{q}_p\}$, and $\{\ddot{q}_p\}$, Eq. (37) can be integrated to determine $\{q_f\}$, and Eq. (38) can be evaluated to determine $\{\lambda(t)\}$.

Linearization of the Unconstrained Motion Equations

In this section we linearize the unconstrained motion equations about a state of steady spin with the tethers of fixed unstrained length. Throughout this section

$$\dot{\underline{\omega}} = \underline{0} \tag{39a}$$

$$\dot{d}_{ij} = 0 \quad i, j = 1, 2, \ldots, N \tag{39b}$$

The coordinates of $\mathcal{P}_i$ relative to $\mathcal{F}_p$ are taken to be

$$r_i = r_i^0 + \Delta r_i(t) \tag{40a}$$

$$\theta_i = \theta_i^0 + \Delta\theta_i(t) \tag{40b}$$

$$z_i = z_i^0 + \Delta z_i(t) \tag{40c}$$

where r_i^0, θ_i^0, and z_i^0 are constants defining the steady state equilibrium configuration. The variables Δr_i, $\Delta\theta_i$, Δz_i and their time derivatives are considered to be quantities of small magnitude, and we will ignore terms in which they appear to second or higher order. Following the notation established in Eqs.(40), we introduce also

$$\underline{u}_i = \underline{u}_i^0 + \underline{\Delta u}_i(t)$$

$$\{q\} = \{q^0\} + \{\Delta q(t)\}$$

Recognizing Eq.(39a) and substituting Eq.(40b) into Eqs.(15) yields

$$\Omega_{1i} \cong \Omega_{1i}^0 - \Omega_{2i}^0\,\Delta\theta_i \tag{41a}$$

$$\Omega_{2i} \cong \Omega_{2i}^0 + \Omega_{1i}^0\,\Delta\theta_i \tag{41b}$$

$$\alpha_{1i} = \alpha_{2i} = 0 \tag{41c}$$

$$\dot{\Omega}_{1i} \cong -\Omega_{2i}^0\,\Delta\dot{\theta}_i \tag{41d}$$

$$\dot{\Omega}_{2i} \cong \Omega_{1i}^0\,\Delta\dot{\theta}_i \tag{41e}$$

Above we have introduced the constants

$$\Omega_{1i}^0 = \omega_x \cos\theta_i^0 + \omega_y \sin\theta_i^0$$

$$\Omega_{2i}^0 = \omega_x \sin\theta_i^0 - \omega_y \cos\theta_i^0$$

Substituting Eqs.(40) and (41) into Eq.(28) and ignoring second and higher order terms, we get

$$\underline{N}_i \cong -[G]_i\,\underline{\Delta\dot{u}}_i - [C]_i\,\underline{\Delta u}_i + \underline{N}_i^0 \tag{42}$$

where

$$\underline{\Delta u}_i = \begin{pmatrix} \Delta r_i \\ \Delta\theta_i \\ \Delta z_i \end{pmatrix} \quad ; \qquad [G]_i = 2m_i \begin{bmatrix} 0 & -\omega_z r_i^0 & -\Omega_{2i}^0 \\ \omega_z r_i^0 & 0 & -\Omega_{1i}^0 r_i^0 \\ \Omega_{2i}^0 & \Omega_{1i}^0 r_i^0 & 0 \end{bmatrix}$$

$$[C]_i = m_i \begin{bmatrix} -[\omega_z^2+(\Omega_{2i}^0)^2] & -\Omega_{2i}^0(2\Omega_{1i}^0 r_i^0+\omega_z z_i^0) & \omega_z \Omega_{1i}^0 \\ -\Omega_{2i}^0(2\Omega_{1i}^0 r_i^0+\omega_z z_i^0) & -r_i^0[(\omega_x^2+\omega_y^2)r_i^0+\omega_z \Omega_{1i}^0 z_i^0] & -\omega_z \Omega_{2i}^0 r_i^0 \\ \omega_z \Omega_{1i}^0 & -\omega_z \Omega_{2i}^0 r_i^0 & -(\omega_x^2+\omega_y^2) \end{bmatrix}$$

$$\underline{N}_i^0 = m_i \begin{pmatrix} [\omega_z^2+(\Omega_{2i}^0)^2]r_i^0 - \omega_z \Omega_{1i}^0 z_i^0 \\ \Omega_{2i}^0(\Omega_{1i}^0 r_i^0+\omega_z z_i^0)r_i^0 \\ -\omega_z \Omega_{1i}^0 r_i^0 + (\omega_x^2+\omega_y^2)z_i^0 \end{pmatrix}$$

The linearized counterpart to the mass matrix appearing in Eq.(27) is

$$[M_0]_i \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & (r_i^0)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{43}$$

Linearizing Eqs.(25) about the steady state solution we have

$$\sigma_{ij} \cong \sigma_{ij}^0 + \sigma_{ij}^1 \tag{44a}$$

$$\tau_{ij} \cong \tau_{ij}^0 + \tau_{ij}^1 \tag{44b}$$

$$\zeta_{ij} \cong \zeta_{ij}^0 + \zeta_{ij}^1 \tag{44c}$$

where the constant terms are defined as

$$\sigma_{ij}^0 = r_j^0 \cos(\theta_j^0-\theta_i^0) - r_i^0 \tag{45a}$$

$$\tau_{ij}^0 = -\tau_{ji}^0 = r_i^0 r_j^0 \sin(\theta_j^0-\theta_i^0) \tag{45b}$$

$$\zeta_{ij}^0 = -\zeta_{ji}^0 = z_j^0 - z_i^0 \tag{45c}$$

and the terms dependent to first order on the deflections from the steady state are recorded as

$$\sigma_{ij}^1 = \cos(\theta_j^0-\theta_i^0)\Delta r_j - \Delta r_i - r_j^0 \sin(\theta_j^0-\theta_i^0)(\Delta\theta_j-\Delta\theta_i) \tag{46a}$$

$$\tau_{ij}^1 = r_i^0 \sin(\theta_j^0-\theta_i^0)\Delta r_j + r_j^0 \sin(\theta_j^0-\theta_i^0)\Delta r_i + r_i^0 r_j^0 \cos(\theta_j^0-\theta_i^0)(\Delta\theta_j-\Delta\theta_i) \tag{46b}$$

$$\zeta_{ij}^1 = \Delta z_j - \Delta z_i \tag{46c}$$

Expanding Eq.(22) to first order we have

$$l_{ij} \cong l_{ij}^0 + l_{ij}^1$$

where

$$l_{ij}^0 = \sqrt{(r_i^0)^2 + (r_j^0)^2 - 2r_i^0 r_j^0 \cos(\theta_j^0 - \theta_i^0) + (z_j^0 - z_i^0)^2}$$

and

$$l_{ij}^1 = \frac{1}{l_{ij}^0}\left[-\sigma_{ji}^0\,\Delta r_j - \sigma_{ij}^0\,\Delta r_i + \tau_{ij}^0(\Delta\theta_j - \Delta\theta_i) + \zeta_{ij}^0(\Delta z_j - \Delta z_i)\right] \tag{47}$$

We also record

$$(l_{ij})^{-1} \cong \frac{1}{l_{ij}^0} - \frac{l_{ij}^1}{(l_{ij}^0)^2} \tag{48}$$

and

$$\dot{l}_{ij} = \dot{l}_{ij}^1 = \frac{1}{l_{ij}^0}\left[-\sigma_{ji}^0\,\Delta\dot{r}_j - \sigma_{ij}^0\,\Delta\dot{r}_i + \tau_{ij}^0(\Delta\dot{\theta}_j - \Delta\dot{\theta}_i) + \zeta_{ij}^0(\Delta\dot{z}_j - \Delta\dot{z}_i)\right] \tag{49}$$

For our stipulated conditions, Eq.(23) simplifies to

$$f_{ij} \cong f_{ij}^0 + k_{ij}\,l_{ij}^1 + c_{ij}\,\dot{l}_{ij}^1 \tag{50}$$

where

$$f_{ij}^0 = f_{ji}^0 = k_{ij}(l_{ij}^0 - d_{ij}) \tag{51}$$

is the steady state tension in the tether connecting $\mathcal{P}_i$ and $\mathcal{P}_j$.

Substituting Eqs.(44),(48) and (50) into Eq.(24), and dropping terms above first order, yields

$$\underline{S}_i \cong \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{f_{ij}^0}{l_{ij}^0}\begin{pmatrix}\sigma_{ij}^0\\ \tau_{ij}^0\\ \zeta_{ij}^0\end{pmatrix}$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{l_{ij}^0}\left\{ f_{ij}^0\begin{pmatrix}\sigma_{ij}^1\\ \tau_{ij}^1\\ \zeta_{ij}^1\end{pmatrix} + \left(k_{ij} - \frac{f_{ij}^0}{l_{ij}^0}\right)\begin{pmatrix}\sigma_{ij}^0\\ \tau_{ij}^0\\ \zeta_{ij}^0\end{pmatrix} l_{ij}^1 \right\} \tag{52}$$

$$+\sum_{\substack{j=1\\j\neq i}}^{N}\frac{c_{ij}}{l_{ij}^{0}}\begin{pmatrix}\sigma_{ij}^{0}\\ \tau_{ij}^{0}\\ \zeta_{ij}^{0}\end{pmatrix}\dot{l}_{ij}^{1}$$

Introducing Eqs.(46),(47) and (49) into Eq.(52), we can write

$$\underline{S}_i \equiv \underline{S}_i^0 - [\mathcal{K}]_i\{\Delta q\} - [\mathcal{D}]_i\{\Delta\dot{q}\} \tag{53}$$

where

$$\underline{S}_i^0 = \sum_{\substack{j=1\\j\neq i}}^{N}\frac{f_{ij}^{0}}{l_{ij}^{0}}\begin{pmatrix}\sigma_{ij}^{0}\\ \tau_{ij}^{0}\\ \zeta_{ij}^{0}\end{pmatrix}$$

$$[\mathcal{K}]_i = -\Big[[K]_{i1}\quad [K]_{i2}\quad \cdots\quad [K]_{iN}\Big] \tag{54}$$

$$[K]_{ij} = \begin{bmatrix}K_{rr} & K_{r\theta} & K_{rz}\\ K_{\theta r} & K_{\theta\theta} & K_{\theta z}\\ K_{zr} & K_{z\theta} & K_{zz}\end{bmatrix}_{ij}$$

Introducing

$$e_{ij}^{0} = e_{ji}^{0} = (l_{ij}^{0})^{-2}\left(k_{ij} - \frac{f_{ij}^{0}}{l_{ij}^{0}}\right)$$

we have for $i \neq j$:

$$(K_{rr})_{ij} = (K_{rr})_{ji} = -e_{ij}^{0}\sigma_{ij}^{0}\sigma_{ji}^{0} + \frac{f_{ij}^{0}}{l_{ij}^{0}}\cos(\theta_j^0-\theta_i^0)$$

$$(K_{r\theta})_{ij} = e_{ij}^{0}\sigma_{ij}^{0}\tau_{ij}^{0} - \frac{f_{ij}^{0}}{l_{ij}^{0}}r_j^0\sin(\theta_j^0-\theta_i^0)$$

$$(K_{rz})_{ij} = e_{ij}^{0}\sigma_{ij}^{0}\zeta_{ij}^{0}$$

$$(K_{\theta r})_{ij} = (K_{r\theta})_{ji} = -e_{ij}^{0}\sigma_{ji}^{0}\tau_{ij}^{0} + \frac{f_{ij}^{0}}{l_{ij}^{0}}r_i^0\sin(\theta_j^0-\theta_i^0)$$

$$(K_{\theta\theta})_{ij} = (K_{\theta\theta})_{ji} = e_{ij}^{0}(\tau_{ij}^{0})^2 + \frac{f_{ij}^{0}}{l_{ij}^{0}}r_i^0 r_j^0\cos(\theta_j^0-\theta_i^0)$$

$$(K_{\theta z})_{ij} = (K_{\theta z})_{ji} = e^0_{ij} \tau^0_{ij} \zeta^0_{ij}$$

$$(K_{zr})_{ij} = (K_{rz})_{ji} = -e^0_{ij} \sigma^0_{ji} \zeta^0_{ij}$$

$$(K_{z\theta})_{ij} = (K_{z\theta})_{ji} = (K_{\theta z})_{ij}$$

$$(K_{zz})_{ij} = (K_{zz})_{ji} = e^0_{ij} (\zeta^0_{ij})^2 + \frac{f^0_{ij}}{l^0_{ij}}$$

For $j = i$ we have:

$$[K]_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{N} \begin{bmatrix} \left(e^0_{ij} (\sigma^0_{ij})^2 + \frac{f^0_{ij}}{l^0_{ij}} \right) & (K_{r\theta})_{ij} & (K_{rz})_{ij} \\ & (K_{\theta\theta})_{ij} & (K_{\theta z})_{ij} \\ \text{sym} & & (K_{zz})_{ij} \end{bmatrix}$$

For the coefficient matrix of the last term on the right hand side of Eq.(53) we have

$$[\mathcal{D}]_i = -\left[[D]_{i1} \ \ [D]_{i2} \ \cdots \ [D]_{iN} \right] \tag{55}$$

$$[D]_{ij} = \frac{c_{ij}}{(l^0_{ij})^2} \begin{bmatrix} -\sigma^0_{ij}\sigma^0_{ji} & \sigma^0_{ij}\tau^0_{ij} & \sigma^0_{ij}\zeta^0_{ij} \\ -\tau^0_{ij}\sigma^0_{ji} & (\tau^0_{ij})^2 & \tau^0_{ij}\zeta^0_{ij} \\ -\zeta^0_{ij}\sigma^0_{ji} & \zeta^0_{ij}\tau^0_{ij} & (\zeta^0_{ij})^2 \end{bmatrix} \qquad i \neq j$$

Evidently $[D]_{ij}$ is not symmetric. We have also

$$[D]_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{c_{ij}}{(l^0_{ij})^2} \begin{bmatrix} (\sigma^0_{ij})^2 & \sigma^0_{ij}\tau^0_{ij} & \sigma^0_{ij}\zeta^0_{ij} \\ & (\tau^0_{ij})^2 & \tau^0_{ij}\zeta^0_{ij} \\ \text{sym} & & (\zeta^0_{ij})^2 \end{bmatrix}$$

Substituting Eqs.(42),(43) and (53) into Eq.(27), and rearranging, yields

$$[M_0]_i \, \underline{\Delta\ddot{u}}_i + [G]_i \, \underline{\Delta\dot{u}}_i + [C]_i \, \underline{\Delta u}_i + [\mathcal{D}]_i \{\Delta\dot{q}\} + [\mathcal{K}]_i \{\Delta q\} \tag{56}$$

$$= \underline{Q}_i^{(e)} + \underline{N}_i^0 + \underline{S}_i^0$$

for $i=1, 2, \ldots, N$. Assembling the system of equations given by Eq.(56) as a single matrix equation, yields

$$[\mathcal{M}_0]\{\Delta\ddot{q}\} + \big([\mathcal{G}] + [\mathcal{D}]\big)\{\Delta\dot{q}\} + \big([\mathcal{C}] + [\mathcal{K}]\big)\{\Delta q\} = \left\{Q^{(e)}\right\} + \{\mathcal{N}_0\} + \{\mathcal{S}_0\} \tag{57}$$

where

$$[\mathcal{M}_0] = \begin{bmatrix} [M_0]_1 & [0] & \cdots & [0] \\ [0] & [M_0]_2 & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots \\ [0] & [0] & \cdots & [M_0]_N \end{bmatrix} \quad ; \quad [\mathcal{G}] = \begin{bmatrix} [G]_1 & [0] & \cdots & [0] \\ [0] & [G]_2 & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots \\ [0] & [0] & \cdots & [G]_N \end{bmatrix}$$

$$[\mathcal{C}] = \begin{bmatrix} [C]_1 & [0] & \cdots & [0] \\ [0] & [C]_2 & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots \\ [0] & [0] & \cdots & [C]_N \end{bmatrix}$$

$$[\mathcal{D}] = \begin{bmatrix} [\mathcal{D}]_1 \\ [\mathcal{D}]_2 \\ \vdots \\ [\mathcal{D}]_N \end{bmatrix} \quad ; \quad [\mathcal{K}] = \begin{bmatrix} [\mathcal{K}]_1 \\ [\mathcal{K}]_2 \\ \vdots \\ [\mathcal{K}]_N \end{bmatrix}$$

$$\{\mathcal{N}_0\} = \begin{pmatrix} \underline{N}_1^0 \\ \underline{N}_2^0 \\ \vdots \\ \underline{N}_N^0 \end{pmatrix} \quad ; \quad \{\mathcal{S}_0\} = \begin{pmatrix} \underline{S}_1^0 \\ \underline{S}_2^0 \\ \vdots \\ \underline{S}_N^0 \end{pmatrix}$$

The vector of external forces appearing in Eq.(57) can be separated into two terms, one constant, $\{Q_0^{(e)}\}$, and the other time dependent, $\{Q_1^{(e)}(t)\}$, i.e.

$$\{Q^{(e)}\} = \{Q_0^{(e)}\} + \{Q_1^{(e)}(t)\} \tag{58}$$

If we consider Eq.(57) under the conditions

$$\{\Delta\ddot{q}\} = \{\Delta\dot{q}\} = \{\Delta q\} = \{Q_1^{(e)}(t)\} = \{0\}$$

then we find the steady state equilibrium equations for the system are

$$\{Q_0^{(e)}\} + \{\mathcal{N}_0\} + \{\mathcal{S}_0\} = \{0\} \tag{59}$$

Equations (59) constitute a system of 3N nonlinear algebraic equations in the quantities; r_i^0, θ_i^0, z_i^0 , $i = 1, 2, \ldots, N$.

Recognizing Eq.(59), the linearized motion equations for the unconstrained system follow from Eq.(57), as

$$[\mathcal{M}_0]\{\Delta\ddot{q}\} + \left([\mathcal{G}] + [\mathcal{D}]\right)\{\Delta\dot{q}\} + \left([\mathcal{C}] + [\mathcal{K}]\right)\{\Delta q\} = \left\{Q_1^{(e)}(t)\right\} \tag{60}$$

The following properties of the coefficient matrices appearing in Eq.(60) are noted:

$$[\mathcal{M}_0] = [\mathcal{M}_0]^T \qquad [\mathcal{G}] = -[\mathcal{G}]^T \qquad [\mathcal{D}] = [\mathcal{D}]^T$$

$$[\mathcal{C}] = [\mathcal{C}]^T \qquad [\mathcal{K}] = [\mathcal{K}]^T$$

Model Singularity

A unique solution to any of the various sets of motion equations derived in this report, (e.g. Eqs.(27), (37), or (60)), requires that the coefficient mass matrix of the generalized accelerations be nonsingular. For the unconstrained systems (linear and nonlinear), the mass matrices $[\mathcal{M}]$, and $[\mathcal{M}_0]$, are diagonal and possess elements r_i and r_i^0, $(i=1,2,\ldots,N)$, respectively. Evidently, if $r_i = 0$ or $r_i^0 = 0$ then the associated mass matrix would be singular. This is a consequence of our choice of coordinates rather than an indication of a physical degeneracy. This deficiency of the models can be avoided by limiting attention to only those conditions for which $r_i \neq 0$ (or $r_i^0 \neq 0$ for linear analyses), $(i=1,2,\ldots,N)$. The situation with the constrained nonlinear equations is somewhat more involved. The generalized mass matrix, from Eq.(37), is $[S_f]^T[\mathcal{M}][S_f]$, and it can be shown that that matrix is nonsingular if $[\mathcal{M}]$ is nonsingular. ($[S_f]^T[\mathcal{M}][S_f]$ may also be nonsingular when $[\mathcal{M}]$ is singular, depending on which degrees of

freedom are specified.) The restriction to conditions for which $r_i \neq 0$, however, guarantees a unique solution for Eq.(37) also.

Kinematics of $\mathcal{F}_p$

The motion $\mathcal{F}_p$ relative to $\mathcal{F}_I$ has been assumed to be a known function of time. To complete the formulation we shall define the orientation of $\mathcal{F}_p$ with respect to $\mathcal{F}_I$ in terms of a general 1-2-3 Euler angle sequence. The orthogonal unit basis vectors $\hat{i}, \hat{j}, \hat{k}$ for each frame referred to below will be denoted with appropriate subscript and superscript identifiers. The signs of all single axis rotations are positive according to the right hand rule. Let $\mathcal{F}_I'$ be an intermediate frame achieved from $\mathcal{F}_I$ by a rotation of angle ψ_1 about the $\hat{I}$ axis. Let frame $\mathcal{F}_p'$ be achieved from $\mathcal{F}_I'$ by rotation of angle ψ_2 about the axis $\hat{j}_p'$. $\mathcal{F}_p$ is achieved from $\mathcal{F}_p'$ by a rotation of angle ψ_3 about the $\hat{k}_p$ axis. The direction cosine matrix transforming vector components from $\mathcal{F}_p$ to $\mathcal{F}_I$ is obtained by concatenating the sequence of single axis rotations yielding

$$[C_{Ip}] = \begin{bmatrix} c\psi_2 c\psi_3 & -c\psi_2 s\psi_3 & s\psi_2 \\ c\psi_1 s\psi_3 + s\psi_1 s\psi_2 c\psi_3 & c\psi_1 c\psi_3 - s\psi_1 s\psi_2 s\psi_3 & -s\psi_1 c\psi_2 \\ s\psi_1 s\psi_3 - c\psi_1 s\psi_2 c\psi_3 & s\psi_1 c\psi_3 + c\psi_1 s\psi_2 s\psi_3 & c\psi_1 c\psi_2 \end{bmatrix}$$

where we have used the shorthand notation: $c\psi_i = \cos\psi_i$, $s\psi_i = \sin\psi_i$. The angular velocity of $\mathcal{F}_p$ with respect to $\mathcal{F}_I$, expressed in terms of the Euler angles is

$$\vec{\omega} = \dot{\psi}_1\,\hat{I} + \dot{\psi}_2\,\hat{j}_p' + \dot{\psi}_3\,\hat{k}_p$$

Resolved into components referred to $\mathcal{F}_p$ we have

$$\underline{\omega} = \begin{bmatrix} c\psi_2 c\psi_3 & s\psi_3 & 0 \\ -c\psi_2 s\psi_3 & c\psi_3 & 0 \\ s\psi_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{bmatrix}$$

$$= [\Pi]\,\underline{\dot{\psi}}$$

Appendix - Inextensible Tethers

In the main body of this report we have developed the equations governing the motion of a system of particles interconnected by idealized extensible tethers subject to applied external forces as well as prescribed motion. The formulation presumes that the length of unstrained tether deployed at any instant is known a-priori. Clearly, the unstrained length of tether deployed has a strong influence on the tether tension, which is to be a principal means of stabilization and control of the system. In certain circumstances (e.g. inverse dynamic analysis) the need to specify tether unstrained length time history proves to be problematical and simplification is obtained by treating the tethers as inextensible. Inextensible tethers are introduced through the definition of inter-particle distance constraints. Unlike extensible tethers, for which tension is described by constitutive equations (as in Eq.(23)), with inextensible tethers the tensions must be solved for as constraint forces. In this appendix all tethers of the system are taken to be inextensible. Supplemental prescribed motion constraints are also considered and the forces necessary to produce those motions are determined.

Let n_t be the total number of tethers in the system, and let tether k connect particles $\mathcal{P}_i$ and $\mathcal{P}_j$. The distance between $\mathcal{P}_i$ and $\mathcal{P}_j$ is given in Eq.(22) by l_{ij}. Let $\overline{d}_{ij}(t)$ be the length of inextensible tether instantaneously deployed between $\mathcal{P}_i$ and $\mathcal{P}_j$. We shall assume that $\overline{d}_{ij}(t)$ is a known function of time. If T_k is the tension in tether k, then the constraint governing tether inextensibility is

$$\begin{cases} l_{ij} = \overline{d}_{ij}(t) & \text{if } T_k > 0 \\ \text{no constraint} & \text{if } T_k \leq 0 \end{cases} \tag{61}$$

Whether the kinematic constraint of Eq.(61) is in effect or not depends on the tension force, so the system is seen to possess a discontinuous number of degrees of freedom. Rather than deal with the complication of the discontinuity, we shall simply enforce the constraint at all times and recognize that the occurrence of non-positive tension invalidates

our results from that point onward. We note that taking Eq.(61) to hold for all T_k corresponds to the condition of a rigid link.

It proves convenient to state the tether constraints in the form

$$h_k(r_i,\theta_i,z_i,r_j,\theta_j,z_j,t) = (l_{ij})^2 - (\bar{d}_{ij})^2 = 0$$

or

$$h_k(\underline{q},t) = r_i^2 + r_j^2 - 2r_ir_j\cos(\theta_j-\theta_i) + (z_j-z_i)^2 - (\bar{d}_{ij})^2 = 0 \qquad (62)$$

For the entire system of n_t tethers the constraints are written as

$$\{h(\underline{q},t)\} = \begin{Bmatrix} h_1(\underline{q},t) \\ h_2(\underline{q},t) \\ \vdots \\ h_{n_t}(\underline{q},t) \end{Bmatrix} = \underline{0} \qquad (63)$$

To permit the study of prescribed motion and to allow the determination of the associated control forces we consider in addition to Eq.(63) the n_p constraints given by Eq.(30). The complete set of n_t+n_p constraints is expressed as

$$\{\Phi(\underline{q},t)\} = \begin{Bmatrix} \{h(\underline{q},t)\} \\ [S_p]^T\{q\} - \{q_p(t)\} \end{Bmatrix} = \underline{0} \qquad (64)$$

It is required that $n_t+n_p \le 3N$, and that the constraints of Eq.(64) be consistent and functionally independent.

Lagranges equations for our system of particles subject to the constraints of Eq.(64) can be written as

$$\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}}\right\} - \left\{\frac{\partial T}{\partial q}\right\} = \left\{Q^{(e)}\right\} - [\Phi_q]^T\{\lambda\} \qquad (65)$$

which must be considered together with the constraint equations given by Eq.(64). Note that in comparison with Eq.(31), Eq.(65) does not contain the generalized forces associated with any internal forces, since here the tether tensions, being constraint forces, do no work in virtual

displacements consistent with the constraints. In Eq.(65) we have introduced the Jacobian of the constraints

$$[\Phi_q] = \begin{bmatrix} [\frac{\partial h}{\partial q}] \\ [S_p]^T \end{bmatrix}$$

where

$$\left[\frac{\partial h}{\partial q}\right] = \begin{bmatrix} \frac{\partial h_1}{\partial r_1} & \frac{\partial h_1}{\partial \theta_1} & \frac{\partial h_1}{\partial z_1} & \frac{\partial h_1}{\partial r_2} & \cdots & \frac{\partial h_1}{\partial z_N} \\ \frac{\partial h_2}{\partial r_1} & \frac{\partial h_2}{\partial \theta_1} & \frac{\partial h_2}{\partial z_1} & \frac{\partial h_2}{\partial r_2} & \cdots & \frac{\partial h_2}{\partial z_N} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial h_{n_t}}{\partial r_1} & \frac{\partial h_{n_t}}{\partial \theta_1} & \frac{\partial h_{n_t}}{\partial z_1} & \frac{\partial h_{n_t}}{\partial r_2} & \cdots & \frac{\partial h_{n_t}}{\partial z_N} \end{bmatrix}$$

Forming the partial derivatives indicated above from Eq.(62) we record

$$\frac{\partial h_k}{\partial r_\ell} = -2(\sigma_{ij}\delta_{i\ell} + \sigma_{ji}\delta_{j\ell})$$

$$\frac{\partial h_k}{\partial \theta_\ell} = 2\tau_{ij}(\delta_{j\ell} - \delta_{i\ell})$$

$$\frac{\partial h_k}{\partial z_\ell} = 2\zeta_{ij}(\delta_{j\ell} - \delta_{i\ell})$$

for $\ell = 1,2,\ldots,N$. δ_{ij} is the usual Kronecker delta symbol. In terms of the notation introduced with Eq.(32), Eq.(65) can be written as

$$[\mathcal{M}]\{\ddot{q}\} + [\Phi_q]^T\{\lambda\} = \{\mathcal{N}\} + \{Q^{(e)}\} \tag{66}$$

One approach to the solution of our system is to form the second time derivative of Eq.(64);

$$[\Phi_q]\{\ddot{q}\} = -\frac{\partial^2}{\partial t^2}\{\Phi\} - \frac{d}{dt}\big([\Phi_q]\big)\{\dot{q}\} \tag{67}$$

where

$$\frac{\partial^2}{\partial t^2}\{\Phi\} = \begin{Bmatrix} \frac{\partial^2}{\partial t^2}\{h\} \\ -\{\ddot{q}_p(t)\} \end{Bmatrix} \quad ; \quad \frac{\partial^2 h_k}{\partial t^2} = -2(\bar{d}_{ij}\ddot{\bar{d}}_{ij} + \dot{\bar{d}}_{ij}^2)$$

and

$$\frac{d}{dt}[\Phi_q] = \begin{bmatrix} \frac{d}{dt}[\frac{\partial h}{\partial q}] \\ [0] \end{bmatrix} \tag{68}$$

The elements of the upper partition of the matrix in Eq.(68) are found to be

$$\frac{d}{dt}\frac{\partial h_k}{\partial r_\ell} = 2\{[r_j(\dot\theta_j-\dot\theta_i)\sin(\theta_j-\theta_i) + \dot r_i - \dot r_j\cos(\theta_j-\theta_i)]\delta_{i\ell} + [r_i(\dot\theta_j-\dot\theta_i)\sin(\theta_j-\theta_i) + \dot r_j - \dot r_i\cos(\theta_j-\theta_i)]\delta_{j\ell}\}$$

$$\frac{d}{dt}\frac{\partial h_k}{\partial \theta_\ell} = 2[(\dot r_i r_j + r_i \dot r_j)\sin(\theta_j-\theta_i) + r_i r_j(\dot\theta_j-\dot\theta_i)\cos(\theta_j-\theta_i)](\delta_{j\ell}-\delta_{i\ell})$$

$$\frac{d}{dt}\frac{\partial h_k}{\partial z_\ell} = 2(\dot z_j - \dot z_i)(\delta_{j\ell}-\delta_{i\ell})$$

Writing Eqs.(66) and (67) together we have

$$\begin{bmatrix} [\mathcal{M}] & [\Phi_q]^T \\ [\Phi_q] & [0] \end{bmatrix}\begin{pmatrix}\{\ddot q\} \\ \{\lambda\}\end{pmatrix} = \begin{pmatrix} \{\mathcal{N}\} + \{Q^{(e)}\} \\ -\frac{\partial^2}{\partial t^2}\{\Phi\} - \frac{d}{dt}[\Phi_q]\{\dot q\}\end{pmatrix} \tag{69}$$

Equations (69) can be solved simultaneously for the acceleration variables and the Lagrange multipliers, to be followed by numerical integration for the generalized coordinates.

<u>Constraint Forces</u>

To calculate the tether tension forces we first establish the corresponding generalized force expressions. As was the case with the extensible tethers, we denote the force acting on $\mathcal{P}_i$ arising from its connection to $\mathcal{P}_j$ by the vector $\vec f_{ij}$, and let f_{ij} denote the scalar tension. Then we can write

$$\vec f_{ij} = \frac{f_{ij}}{l_{ij}}\vec l_{ij}$$

For a tether in tension $f_{ij}>0$, while $f_{ij}<0$ indicates compression. If $\mathcal{P}_i$ and $\mathcal{P}_j$ are not connected, $f_{ij}=0$. Unlike the situation for extensible tethers, where f_{ij} was expressible as an explicit function of the system state variables, here the terms f_{ij} are unknowns. The total force on $\mathcal{P}_i$ from all its tether connections is, again,

$$\vec{f}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \vec{f}_{ij}$$

The above expression is identical to that obtained for extensible tethers. The virtual work performed by the tether forces is again given by Eq.(19), which leads to the generalized forces given in Eq.(24). Here we rewrite Eq.(24) as

$$\underline{Q}_i^{(i)} = [\, \underline{\Gamma}_{i1} \;\; \underline{\Gamma}_{i2} \;\cdots\; \underline{\Gamma}_{iN} \,] \begin{pmatrix} f_{i1} \\ f_{i2} \\ \vdots \\ f_{iN} \end{pmatrix} = [\,\Gamma\,]_i \begin{pmatrix} f_{i1} \\ f_{i2} \\ \vdots \\ f_{iN} \end{pmatrix} \tag{70}$$

where

$$\underline{\Gamma}_{ij} = \frac{1}{l_{ij}} \begin{pmatrix} \sigma_{ij} \\ \tau_{ij} \\ \zeta_{ij} \end{pmatrix}$$

Introducing the vector of tether tensions; $\{T\}^T = \{T_1 \;\; T_2 \;\cdots\; T_{n_t}\}$, we can write

$$\begin{pmatrix} f_{i1} \\ f_{i2} \\ \vdots \\ f_{iN} \end{pmatrix} = [\,B\,]_i \{T\} \tag{71}$$

where $[B]_i$ is a Boolean selection matrix that establishes the correspondence between particle pairs and their associated tethers. Substituting Eq.(71) into (70) we obtain

$$\underline{Q}_i^{(i)} = [\,\Gamma\,]_i [\,B\,]_i \{T\} = [\,\Delta\,]_i \{T\}$$

The vector of generalized forces associated with the tether tensions for the entire system is assembled as

$$\{Q^{(i)}\} = \begin{pmatrix} \underline{Q}_1^{(i)} \\ \underline{Q}_2^{(i)} \\ \vdots \\ \underline{Q}_N^{(i)} \end{pmatrix} = \begin{bmatrix} [\,\Delta\,]_1 \\ [\,\Delta\,]_2 \\ \\ [\,\Delta\,]_N \end{bmatrix} \{T\} = [\,\Delta\,]\{T\} \tag{72}$$

In addition to the tether tension forces, we consider external constraint forces to act on each particle to ensure satisfaction of the prescribed motion constraints. These forces may be thought to arise from thrusters. As was shown in the development leading to Eq. (34), and continuing the notation established there, the generalized forces associated with external constraint forces can be expressed as

$$\{F^{(c)}\}^T = \left(F_{r1}^{(c)} \quad r_1 F_{t1}^{(c)} \quad F_{z1}^{(c)} \,\vdots\, F_{r2}^{(c)} \quad r_2 F_{t2}^{(c)} \quad F_{z2}^{(c)} \,\vdots\, \cdots \right) \tag{73}$$

It is known from analytical mechanics theory that the generalized forces associated with the constraints of Eq. (64) are given by

$$\{Q^{(c)}\} = -[\Phi_q]^T\{\lambda\} \tag{74}$$

The generalized constraint forces of Eq. (74) arise from the inextensible tethers and the external particle constraint forces needed to enforce any additional prescribed motion conditions. In terms of Eqs. (72) - (74) we write

$$-[\Phi_q]^T\{\lambda\} = [\Delta]\{T\} + \{F^{(c)}\} \tag{75}$$

We are interested in finding the tether tension forces that most closely satisfy the left hand side of Eq. (75), i.e. we wish to solve the overdetermined system

$$[\Delta]\{T\} = -[\Phi_q]^T\{\lambda\}$$

The Moore-Penrose generalized inverse[5] provides us with the desired solution

$$\{T\} = -[\Delta]^+[\Phi_q]^T\{\lambda\} \tag{76}$$

Substituting Eq. (76) into (75) yields a system of particle forces sufficient to complete satisfaction of the constraints

$$\{F^{(c)}\} = ([\Delta][\Delta]^+ - [1])[\Phi_q]^T\{\lambda\}$$

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